Spin Factor in Path Integral Representation for Dirac Propagator in External Fields

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(February 1, 2008)

arXiv:hep-th/9608179v1 27 Aug 1996

Abstract

We study the spin factor problem both in 3+1 and 2+1 dimensions which are essentially different for spin factor construction. Doing all Grassmann integrations in the corresponding path integral representations for Dirac propagator we get representations with spin factor in arbitrary external field. Thus, the propagator appears to be presented by means of bosonic path integral only. In $3 + 1$ dimensions we present a simple derivation of spin factor avoiding some unnecessary steps in the original brief letter (Gitman, Shvartsman, Phys. Lett. **B318** (1993) 122) which themselves need some additional justification. In this way the meaning of the surprising possibility of complete integration over Grassmann variables gets clear. In $2 + 1$ dimensions the derivation of the spin factor is completely original. Then we use the representations with spin factor for calculations of the propagator in some configurations of ex-

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ternal fields. Namely, in constant uniform electromagnetic field and in its combination with a plane wave field.

I. INTRODUCTION

Propagators of relativistic particles in external fields (electromagnetic, non-Abelian or gravitational) contain important information about the quantum behavior of these particles. Moreover, if such propagators are known in arbitrary external field one can find exact one-particle Green functions in the corresponding quantum field theory taking functional integrals over all external fields. Dirac propagator in an external electromagnetic field distinguishes from that of a scalar particle by a complicated spinor structure. The problem of its path integral representation has attracted researchers' attention already for a long time. Thus, Feynman who has written first his path integral for the probability amplitude in nonrelativistic quantum mechanics [1] and then wrote a path integral for the causal Green function of Klein-Gordon equation (scalar particle propagator) [2], had also made an attempt to derive a representation for Dirac propagator via a bosonic path integral [3]. After the introduction of the integral over Grassmann variables by Berezin it turned to be possible to present this propagator via both bosonic and Grassmann variables, the latter describe spinning degrees of freedom. Representations of this kind have been discussed in the literature for a long time in different contexts [4]. Nevertheless, attempts to write Dirac propagator via only a bosonic path integral continued. Thus, Polyakov [6] assumed that the propagator of a free Dirac electron in $D = 3$ Euclidean space-time can be presented by means of a bosonic path integral similar to the scalar particle case, modified by so called spin factor (SF). This idea was developed in [7] e.g. to write SF for Dirac fermions, interacting with a non-Abelian gauge field in D-dimensional Euclidean space-time. In those representations SF itself was presented via some additional bosonic path integrals and its γ-matrix structure was not defined explicitly. Surprisingly, it was shown in [9] that all Grassmann integrations in the representation of Dirac propagator in an arbitrary external field in $3 + 1$ dimensions can be done, so that an expression for SF was derived as a given functional of the bosonic trajectory. Having such representation with SF, one can use it to calculate the propagator in some particular cases of external fields. This way of calculation provides automatically

an explicit spinor structure of the propagators which can be used for concrete calculations in the Furry picture (see for example, [10,11]).

In the recent work [13] the propagator of a spinning particle in an external field was presented via a path integral in arbitrary dimensions. It turns out that the problem has different solutions in even and odd dimensions. In even dimensions the representation is just a generalization of one in four dimensions mentioned above. In odd dimensions the solution was presented for the first time and differs essentially from the even-dimensional case. Using the representation in odd dimensions one can derive an expression for SF doing Grassmann integrations similar to the four-dimensional case [9].

In the present paper we continue the consideration of the problems related to the SF conception. Namely, we discuss derivation of SF both in even and odd dimensions on the examples of $3 + 1$ and $2 + 1$ cases and then we use the path integral representations with SF to calculate the propagators in some configurations of external fields. In $3 + 1$ dimensions we present a simple derivation of SF avoiding some unnecessary steps in the original brief letter [9] which themselves needed some additional justification. In this way the meaning of the surprising possibility of complete integration over Grassmann variables gets clear. Then we use the representation with SF for calculations of the propagator in a constant uniform electromagnetic field and its combination with a plane wave. Due to the fact that this way of calculations provides automatically an explicit γ-matrix structure of the propagator, the representations obtained differ from those found by means of other methods, for example differs from the well known Schwinger formula in the constant uniform electromagnetic field. To compare both representations we prove in the Appendix some complicated decompositions of functions on the γ matrices. In $2+1$ dimensions the derivation of SF is completely original. We calculate then the propagator in these dimensions in constant electromagnetic field by means of the representation with SF. The result is new and cannot be derived from the $3 + 1$-dimensional case by means of a dimensional reduction.

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II. SPIN FACTOR IN $3+1$ DIMENSIONS

A. Doing integrals over Grassmann variables

The propagator of a relativistic spinning particle in an external electromagnetic field $A_\mu(x)$ is the causal Green function $S^c(x,y)$ of the Dirac equation in this field,

$$[\gamma^\mu (i\partial_\mu - gA_\mu) - m] S^c(x,y) = -\delta^4(x-y) , \qquad (1)$$

where $x = (x^\mu)$, $[\gamma^\mu , \gamma^\nu]_+ = 2\eta^{\mu\nu}$; $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$; $\mu,\nu = 0,1,2,3$.

In the paper [8] the following Lagrangian path integral representation for the propagator was obtained in $3+1$ dimensions,

$$S^c = S^c(x_{out}, x_{in}) = -\tilde{S}^c \gamma^5 , \qquad (2)$$

$$\tilde{S}^c = \exp\left\{i\tilde{\gamma}^n \frac{\partial_\ell}{\partial\theta^n}\right\} \int_0^\infty de_0 \int d\chi_0 \int_{e_0} M(e)De \int_{\chi_0} D\chi \int_{x_{in}}^{x_{out}} Dx \int D\pi_e \int D\pi_\chi$$

$$\times \int_{\psi(0)+\psi(1)=\theta} D\psi \exp\left\{i\int_0^1 \left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}^\mu A_\mu + iegF_{\mu\nu}\psi^\mu\psi^\nu\right.\right.$$

$$\left.\left.+i\left(\frac{\dot{x}_\mu\psi^\mu}{e} - m\psi^5\right)\chi - i\psi_n\dot{\psi}^n + \pi_e\dot{e} + \pi_\chi\dot{\chi}\right]d\tau + \psi_n(1)\psi^n(0)\right\}\Bigg|_{\theta=0} ,$$

where $\tilde{\gamma}^\mu = \gamma^5\gamma^\mu$, $\tilde{\gamma}^5 = \tilde{\gamma}^0\tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3 = \gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$, $[\gamma^m , \gamma^n]_+ = 2\eta^{mn}$, $m,n = 0,1,2,3,5$, $\eta^{mn} = \mathrm{diag}(1,-1,-1,-1,-1)$; θ^n are auxiliary Grassmann (odd) variables, anticommuting by definition with the γ-matrices; $x^\mu(\tau)$, $e(\tau)$, $\pi_e(\tau)$ are bosonic trajectories of integration; $\psi^n(\tau)$, $\chi(\tau)$, $\pi_\chi(\tau)$ are odd trajectories of integration; boundary conditions

$$x(0) = x_{in}, \ x(1) = x_{out}, \ e(0) = e_0, \ \psi^n(0) + \psi^n(1) = \theta^n, \ \chi(0) = \chi_0$$

take place; the measure $M(e)$ and $D\psi$ have the form

$$M(e) = \int Dp \exp\left\{\frac{i}{2}\int_0^1 ep^2 d\tau\right\}, \ D\psi = D\psi \left[\int_{\psi(0)+\psi(1)=0} D\psi \exp\left\{\int_0^1 \psi_n\dot{\psi}^n d\tau\right\}\right]^{-1}, \quad (3)$$

and $\frac{\partial_\ell}{\partial\theta^n}$ stands for the left derivatives. Let us demonstrate that the propagator (2) can be expressed only through a bosonic path integral over the coordinates x. For this purpose one needs to perform several functional integrations, in particular, to fulfil all the Grassmann

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integrations. First, one can integrate over π_e and π_χ, and then using the arising δ-functions to remove the functional integration over e and χ,

$$\tilde{S}^c = -\exp\left\{i\tilde{\gamma}^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0\, M(e_0)\int_{x_{in}}^{x_{out}} Dx\int_{\psi(0)+\psi(1)=\theta}\mathcal{D}\psi\int_0^1\left(\frac{\dot{x}_\mu\psi^\mu}{e_0}-m\psi^5\right)d\tau$$
$$\times\exp\left\{i\int_0^1\left[-\frac{\dot{x}^2}{2e_0}-\frac{e_0}{2}m^2-g\dot{x}^\mu A_\mu+ige_0 F_{\mu\nu}\psi^\mu\psi^\nu-i\psi_n\dot{\psi}^n\right]d\tau+\psi_n(1)\psi^n(0)\right\}\Bigg|_{\theta=0}.$$

Then, changing the integration variables,

$$\psi^n = \frac{1}{2}\left(\xi^n+\theta^n\right), \tag{4}$$

and introducing odd sources $\rho_n(\tau)$ for the new variables $\xi^n(\tau)$, we get

$$\tilde{S}^c = -\frac{1}{2}\exp\left\{i\tilde{\gamma}^n\frac{\partial_\ell}{\partial\theta^n}\right\}\int_0^\infty de_0\, M(e_0)\int_{x_{in}}^{x_{out}} Dx\,\exp\left\{-i\left[\frac{\dot{x}_\mu\star\dot{x}^\mu}{2e_0}+\frac{e_0}{2}m^2+g\dot{x}^\mu\star A_\mu\right]\right.$$
$$\left.-\frac{ge_0}{4}\theta^\mu\star\mathcal{F}_{\mu\nu}\star\theta^\nu\right\}\left[\frac{\dot{x}_\mu}{e_0}\star\left(\frac{\delta_\ell}{\delta\rho_\mu}+\theta^\mu\right)-m\star\left(\frac{\delta_\ell}{\delta\rho_5}+\theta^5\right)\right]R[x,\rho,\theta]\Bigg|_{\rho=0,\theta=0}, \tag{5}$$

where

$$R[x,\rho,\theta] = \int_{\xi(0)+\xi(1)=0}\mathcal{D}\xi\,\exp\left\{\frac{1}{4}\xi_n\star\dot{\xi}^n\right.$$
$$\left.-\frac{ge_0}{4}\xi^\mu\star\mathcal{F}_{\mu\nu}\star\xi^\nu-\frac{ge_0}{2}\theta^\mu\star\mathcal{F}_{\mu\nu}\star\xi^\nu+\rho_n\star\xi^n\right\}, \tag{6}$$

$$\mathcal{D}\xi = D\xi\left[\int_{\xi(0)+\xi(1)=0}D\xi\,\exp\left\{\frac{1}{4}\xi_n\star\dot{\xi}^n\right\}\right]^{-1}. \tag{7}$$

Here condensed notations are used in which $\mathcal{F}_{\mu\nu}$ is understood as a matrix with continuous indices,

$$\mathcal{F}_{\mu\nu}(\tau,\tau') = F_{\mu\nu}(x(\tau))\delta(\tau-\tau'), \tag{8}$$

and integration over τ is denoted by star, e.g.

$$\xi_n\star\dot{\xi}^n = \int_0^1\xi_n(\tau)\dot{\xi}^n(\tau)d\tau.$$

Sometimes discrete indices will be also omitted. In this case all tensors of second rank have to be understood as matrices with lines marked by the first contravariant indices of the tensors, and with columns marked by the second covariant indices of the tensors.

The Grassmann Gaussian path integral in (6) can be evaluated straightforwardly [14] to be

$$R[x, \rho, \theta] = \left\{ \mathrm{Det} \left[\mathcal{U}^{-1}(0) \mathcal{U}(g) \right] \right\}^{1/2} \exp \left\{ J^m \star \mathcal{W}_{mn} \star J^n \right\}, \tag{9}$$

where the matrices $\mathcal{W}(g)$ and $\mathcal{U}(g)$ have the form

$$\mathcal{W}_{mn}(g) = \begin{pmatrix} \mathcal{U}_{\mu\nu}(g) & 0 \\ 0 & -\delta'(\tau - \tau') \end{pmatrix},$$

$$\mathcal{U}_{\mu\nu}(g) = \eta_{\mu\nu} \delta'(\tau - \tau') - g e_0 \mathcal{F}_{\mu\nu}(\tau, \tau'), \tag{10}$$

and

$$J_\mu = \rho_\mu + \frac{g e_0}{2} \mathcal{F}_{\mu\nu} \star \theta^\nu, \qquad J_5 = \rho_5.$$

The determinant in (9) should be understood as

$$\mathrm{Det} \left[\mathcal{U}^{-1}(0) \mathcal{U}(g) \right] = \exp \mathrm{Tr} \left[\log \mathcal{U}(g) - \log \mathcal{U}(0) \right]$$

$$= \exp \left\{ -e_0 \mathrm{Tr} \int_0^g dg' \mathcal{R}(g') \star \mathcal{F} \right\}, \tag{11}$$

where $\mathcal{R}(g)$ is the inverse to $\mathcal{U}(g)$, considered as an operator acting in the space of the antiperiodic functions,

$$\frac{d}{d\tau} \mathcal{R}_{\mu\nu}(g|\tau, \tau') - g e_0 F_\mu{}^\lambda(x(\tau)) \mathcal{R}_{\lambda\nu}(g|\tau, \tau') = \eta_{\mu\nu} \delta(\tau - \tau'),$$

$$\mathcal{R}_{\mu\nu}(g|1, \tau) = -\mathcal{R}_{\mu\nu}(g|0, \tau), \qquad \forall \tau \in (0, 1). \tag{12}$$

Substituting (9) and (11) into (5) and performing then the functional differentiations with respect to ρ_n, we get

$$\tilde{S}^c = -\frac{1}{2} \exp \left\{ i\tilde{\gamma}^n \frac{\partial_\ell}{\partial \theta^n} \right\} \int_0^\infty de_0 M(e_0) \int_{x_{in}}^{x_{out}} Dx \exp \left\{ -\frac{i}{2} \left[\frac{\dot{x}^\mu \star \dot{x}^\mu}{e_0} + e_0 m^2 \right. \right.$$

$$+ g \dot{x}^\mu A_\mu] \right\} \left[\frac{\dot{x}^\mu}{e_0} \star K_{\mu\nu} \theta^\nu - m\theta^5 \right] \left[1 - \frac{g e_0}{4} B_{\alpha\beta} \theta^\alpha \theta^\beta + \frac{g^2 e_0^2}{16} B_{\alpha\beta} B^{*\alpha\beta} \theta^0 \theta^1 \theta^2 \theta^3 \right]$$

$$\times \exp \left\{ -\frac{e_0}{2} \int_0^\infty dg' \mathrm{Tr} \mathcal{R}(g') \star \mathcal{F} \right\} \bigg|_{\theta=0}, \tag{13}$$

where following notations are used,

$$B_{\mu\nu} = F_{\mu\lambda} \star K^{\lambda}{}_{\nu}, \quad B^{*\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}B_{\alpha\beta}, \quad K_{\mu\nu} = \eta_{\mu\nu} + ge_0\mathcal{R}_{\mu\lambda}(g) \star F^{\lambda}{}_{\nu}, \tag{14}$$

and $\epsilon^{\mu\nu\alpha\beta}$ is Levi-Civita symbol normalized by $\epsilon^{0123} = 1$.

Differentiation with respect to θ^n in (13) replaces the products of the variables θ^n by the corresponding antisymmetrized products of the matrices $i\tilde{\gamma}^n$. Finally, passing to the propagator S^c and using the identities

$$\gamma^{[\lambda}\gamma^{\mu}\gamma^{\nu]} = \gamma^{\lambda}\gamma^{[\mu}\gamma^{\nu]} - 2\eta^{\lambda[\mu}\gamma^{\nu]}, \qquad \sigma^{\mu\nu} = i\gamma^{[\mu}\gamma^{\nu]}, \tag{15}$$

where antisymmetrization over the corresponding sets of indices is denoted by brackets, one gets

$$S^c(x_{out}, x_{in}) = \frac{i}{2}\int_0^{\infty} de_0 \int_{x_{in}}^{x_{out}} Dx\, M(e_0)\Phi[x, e_0]\, \exp\{iI[x, e_0]\}\,, \tag{16}$$

where $I[x, e_0]$ is the action of a relativistic spinless particle,

$$I[x, e_0] = -\int_0^1 \left[\frac{\dot{x}^2}{2e_0} + \frac{e_0}{2}m^2 + g\dot{x}A(x)\right] d\tau\,, \tag{17}$$

and $\Phi[x, e_0]$ is SF,

$$\Phi[x,\, e_0] = \left[m + (2e_0)^{-1}\dot{x}^{\mu} \star K_{\mu\lambda}\left(2\eta^{\lambda\kappa} - ge_0 B^{\lambda\kappa}\right)\gamma_{\kappa}\right.$$
$$\left. -\frac{ig}{4}\left(me_0 + \dot{x}^{\mu} \star K_{\mu\lambda}\gamma^{\lambda}\right)B_{\kappa\nu}\sigma^{\kappa\nu} + m\frac{g^2 e_0^2}{16}B^*_{\alpha\beta}B^{\alpha\beta}\gamma^5\right]\exp\left\{-\frac{e_0}{2}\int_0^g dg'\,\mathrm{Tr}\,Q(g') \star \mathcal{F}\right\}. \tag{18}$$

B. Propagator in constant uniform electromagnetic field

In the case of a constant uniform field $(F_{\mu\nu} = const)$, which we are going to discuss in this section, the functionals $\mathcal{R}$, K and B do not depend on the trajectory x and can be calculated straightforwardly,

$$\mathcal{R}(g) = \frac{1}{2}\left(\eta\varepsilon(\tau - \tau') - \tanh\frac{ge_0 F}{2}\right)\exp\{e_0 gF(\tau - \tau')\}\,,$$
$$K = \left(\eta - \tanh\frac{ge_0 F}{2}\right)\exp\left(ge_0 F\tau\right)\,, \qquad B = \frac{2}{ge_0}\tanh\frac{ge_0 F}{2}\,. \tag{19}$$

Using them in (18) and integrating over τ whenever possible, we obtain SF in the constant uniform field,

$$
\begin{aligned}
\Phi[x,\,e_0] = \left(\det\cosh\frac{ge_0F}{2}\right)^{1/2} & \left\{ m\left[1 - \frac{i}{2}\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right.\right. \\
& + \frac{1}{4}\left(\tanh\frac{ge_0F}{2}\right)^*_{\mu\nu}\left(\tanh\frac{ge_0F}{2}\right)^{\mu\nu}\gamma^5\right] + \frac{1}{e_0}\left(\int_0^1 \dot{x}\exp(ge_0F\tau)d\tau\right) \\
& \left.\times\left(\eta - \tanh\frac{ge_0F}{2}\right)\left[\left(\eta - \tanh\frac{ge_0F}{2}\right)\gamma - \frac{i}{2}\gamma\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right]\right\}.
\end{aligned}
\tag{20}
$$

We can see that in the field under consideration SF is linear in the trajectory $x^\mu(\tau)$. That facilitates the bosonic integration in the expression (16).

In spite of the fact that SF is a gauge invariant object, the total propagator is not. It is clear from the expression (16) where one needs to choose a particular gauge for the potentials A_μ. Namely, we are going to use the following potentials

$$
A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu,
\tag{21}
$$

for the constant uniform field $F_{\mu\nu} = const$. Thus, one can see that the path integral (16) is quasi-Gaussian in the case under consideration. Let us make there the shift $x \to y + x_{cl}$, with x_{cl} a solution of the classical equations of motion

$$
\frac{\delta I}{\delta x} = 0 \quad \Leftrightarrow \quad \ddot{x}_\mu - ge_0F_{\mu\nu}\dot{x}^\nu = 0
\tag{22}
$$

subjected to the boundary conditions $x_{cl}(0) = x_{in}$, $x_{cl}(1) = x_{out}$. Then the new trajectories of integration y obey zero boundary conditions, $y(0) = y(1) = 0$. Due to the quadratic structure of the action $I[x, e_0]$ and the linearity of SF in x one can make the following substitutions in the path integral:

$$
\begin{aligned}
I[y + x_{cl},\, e_0] &\to I[x_{cl},\, e_0] + I[y,\, e_0] + \frac{e_0}{2}m^2, \\
\Phi[y + x_{cl},\, e_0] &\to \Phi[x_{cl},\, e_0] = \Psi(x_{out},\, x_{in},\, e_0).
\end{aligned}
\tag{23}
$$

Doing also a convenient replacement of variables $p \to \frac{p}{\sqrt{e_0}}$, $y \to y\sqrt{e_0}$, we get

$$
\begin{aligned}
S^c = \frac{i}{2}\int_0^\infty \frac{de_0}{e_0^2}\, &\Psi(x_{out},\, x_{in},\, e_0)\, e^{iI[x_{cl},\,e_0]} \\
&\times\int_0^0 Dy\int Dp\exp\left\{\frac{i}{2}\int_0^1\left(p^2 - \dot{y}^2 - ge_0yF\dot{y}\right)d\tau\right\}.
\end{aligned}
\tag{24}
$$

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One can see that the path integral in (24) is, in fact, the kernel of the Klein-Gordon propagator in the proper-time representation. This path integral can be presented as

$$\int_0^0 Dy \int Dp \exp\left\{\frac{i}{2}\int_0^1 \left(p^2 - \dot{y}^2 - ge_0 y F \dot{y}\right) d\tau\right\} =$$
$$\left[\frac{\text{Det}\left(\eta_{\mu\nu}\partial_\tau^2 - ge_0 F_{\mu\nu}\partial_\tau\right)}{\text{Det}\left(\eta_{\mu\nu}\partial_\tau^2\right)}\right]^{-1/2} \int_0^0 Dy \int Dp \exp\left\{\frac{i}{2}\int_0^1 \left(p^2 - \dot{y}^2\right) d\tau\right\}.$$

Cancelling the factor $\text{Det}\left(-\eta_{\mu\nu}\right)$ in the ratio of the determinants one obtains

$$\frac{\text{Det}\left(\eta_{\mu\nu}\partial_\tau^2 - ge_0 F_{\mu\nu}\partial_\tau\right)}{\text{Det}\left(\eta_{\mu\nu}\partial_\tau^2\right)} = \frac{\text{Det}\left(-\delta_\nu^\mu \partial_\tau^2 + ge_0 F^\mu{}_\nu \partial_\tau\right)}{\text{Det}\left(-\delta_\nu^\mu \partial_\tau^2\right)}. \tag{25}$$

One can also make the replacement

$$- \mathbf{I}\partial_\tau^2 + ge_0 F \partial_\tau \rightarrow -\mathbf{I}\partial_\tau^2 + \frac{g^2 e_0^2}{4} F^2, \tag{26}$$

where $\mathbf{I}$ stands for the unit $4{\times}4$ matrix, in the RHS of (25) because the spectra of both operators coincide. Indeed,

$$- \mathbf{I}\partial_\tau^2 + ge_0 F \partial_\tau = \exp\left(\frac{ge_0}{2}F\tau\right)\left(-\mathbf{I}\partial_\tau^2 + \frac{g^2 e_0^2}{4}F^2\right)\exp\left(-\frac{ge_0}{2}F\tau\right), \tag{27}$$

and zero boundary conditions are invariant under the transformation $y \rightarrow \exp(\frac{ge_0 F\tau}{2})y$. Then, using (26) and the value of the free path integral [8],

$$\frac{i}{2}\int_0^0 Dy \int Dp \exp\{\frac{i}{2}\int d\tau(p^2 - \dot{y}^2)\} = \frac{1}{8\pi^2},$$

related, in fact, to the definition of the measure, we obtain

$$S^c = \frac{1}{8\pi^2}\int_0^\infty \frac{de_0}{e_0^2}\Psi(x_{out},\, x_{in},\, e_0)\, e^{iI[x_{cl},\, e_0]}\left[\frac{\text{Det}\left(-\mathbf{I}\partial_\tau^2 + \frac{g^2 e_0^2}{4}F^2\right)}{\text{Det}\left(-\mathbf{I}\partial_\tau^2\right)}\right]^{-1/2}. \tag{28}$$

The ratio of the determinants can be written now as

$$\frac{\text{Det}\left(-\mathbf{I}\partial_\tau^2 + \frac{g^2 e_0^2}{4}F^2\right)}{\text{Det}\left(-\mathbf{I}\partial_\tau^2\right)} = \exp \text{Tr}\left[\ln\left(-\mathbf{I}\partial_\tau^2 + \frac{g^2 e_0^2}{4}F^2\right) - \ln\left(-\mathbf{I}\partial_\tau^2\right)\right]$$
$$= \exp \text{Tr}\left[\frac{e_0^2}{2}F^2 \int_0^g d\lambda\,\lambda\left(-\mathbf{I}\partial_\tau^2 + \frac{\lambda^2 e_0^2}{4}F^2\right)^{-1}\right]$$
$$= \exp \text{tr}\left[\frac{e_0^2}{2}F^2 \int_0^g d\lambda\,\lambda\sum_{n=1}^\infty\left(\pi^2 n^2 \mathbf{I} + \frac{\lambda^2 e_0^2}{4}F^2\right)^{-1}\right]. \tag{29}$$

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The trace in the infinite-dimensional space in the second line of eq.(29) is taken and only one in the 4-dimensional space remains. Using the formula

$$\sum_{n=1}^{\infty} \left(\pi^2 n^2 + \kappa^2\right)^{-1} = \frac{1}{2\kappa} \coth \kappa - \frac{1}{2\kappa^2},$$

which is also valid if κ is an arbitrary 4×4 matrix, and integrating in (29), we find

$$\frac{\text{Det}\left(-\mathbf{I}\partial_\tau^2 + \frac{g^2 e_0^2}{4} F^2\right)}{\text{Det}\left(-\mathbf{I}\partial_\tau^2\right)} = \det\left(\frac{\sinh \frac{g e_0 F}{2}}{\frac{g e_0 F}{2}}\right). \tag{30}$$

Thus,

$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{e_0 g F}{2}}{gF}\right)^{-\frac{1}{2}} \Psi(x_{out},\, x_{in},\, e_0) e^{iI[x_{cl}, e_0]}, \tag{31}$$

where the function $\Psi(x_{out},\, x_{in},\, e_0)$ is SF on the classical trajectory x_{cl}. The latter can be easily found solving the eq.(22):

$$x_{cl} = (\exp(g e_0 F) - \eta)^{-1} \left[\exp(g e_0 F \tau)(x_{out} - x_{in}) + \exp(g e_0 F) x_{in} - x_{out}\right]. \tag{32}$$

Substituting (32) into eqs.(23) and (31), we obtain

$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{g e_0 F}{2}}{gF}\right)^{-\frac{1}{2}} \Psi(x_{out},\, x_{in},\, e_0)$$
$$\times \exp\left\{\frac{ig}{2} x_{out} F x_{in} - \frac{i}{2} e_0 m^2 - \frac{ig}{4}(x_{out} - x_{in}) F \coth\left(\frac{g e_0 F}{2}\right)(x_{out} - x_{in})\right\}. \tag{33}$$

where

$$\Psi(x_{out}, x_{in}, e_0) = \left[m + \frac{g}{2}(x_{out} - x_{in}) F \left(\coth \frac{g e_0 F}{2} - 1\right)\gamma\right]$$
$$\times \sqrt{\det \cosh \frac{g e_0 F}{2}} \left[1 - \frac{i}{2}\left(\tanh \frac{g e_0 F}{2}\right)_{\mu\nu} \sigma^{\mu\nu}\right.$$
$$+ \left.\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\left(\tanh \frac{g e_0 F}{2}\right)_{\alpha\beta}\left(\tanh \frac{g e_0 F}{2}\right)_{\mu\nu}\gamma^5\right]. \tag{34}$$

Now we are going to compare the representation (33) with the Schwinger formula [15], which he has been derived in the same case of constant field by means of the proper-time method. The Schwinger representation has the form

$$S^c(x_{out}, x_{in}) = \frac{1}{32\pi^2} \left[\gamma^\mu \left(i\frac{\partial}{\partial x_{out}^\mu} - gA_\mu(x_{out}) \right) + m \right] \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF} \right)^{-1/2}$$

$$\times \exp \left\{ \frac{i}{2} \left[gx_{out}Fx_{in} - e_0 m^2 - (x_{out} - x_{in})\frac{gF}{2} \coth \frac{ge_0 F}{2} (x_{out} - x_{in}) - \frac{ge_0}{2} F_{\mu\nu}\sigma^{\mu\nu} \right] \right\}. \quad (35)$$

Doing the differentiation with respect to x_{out}^μ we transform the formula (35) to a form which is convenient for the comparison with our representation (31),

$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF} \right)^{-1/2} \Psi_S(x_{out}, x_{in}, e_0)$$

$$\times \exp \left\{ i\frac{g}{2} x_{out}Fx_{in} - \frac{i}{2}e_0 m^2 - i\frac{g}{4}(x_{out} - x_{in})F \coth \left(\frac{ge_0 F}{2} \right)(x_{out} - x_{in}) \right\}, \quad (36)$$

where the function Ψ_S is given by

$$\Psi_S(x_{out}, x_{in}, e_0) = \left[m + \frac{g}{2}(x_{out} - x_{in})F(\coth \frac{ge_0 F}{2} - 1)\gamma \right] \exp(-i\frac{e_0 g}{4} F_{\mu\nu}\sigma^{\mu\nu}). \quad (37)$$

Thus one needs only to compare the functions Ψ and Ψ_S. They coincide, since the following formula takes place (see Appendix B), where $\omega_{\mu\nu}$ is an arbitrary antisymmetric tensor,

$$\exp \left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} \right) = \sqrt{\det \cosh \frac{\omega}{2}} \left[1 - \frac{i}{2} \left(\tanh \frac{\omega}{2} \right)_{\mu\nu} \sigma^{\mu\nu} \right.$$

$$\left. + \frac{1}{8}\epsilon^{\alpha\beta\mu\nu} \left(\tanh \frac{\omega}{2} \right)_{\alpha\beta} \left(\tanh \frac{\omega}{2} \right)_{\mu\nu} \gamma^5 \right]. \quad (38)$$

In fact, the latter formula presents a linear decomposition of a finite Lorentz transformation in the independent γ-matrix structures.

C. Propagator in a constant uniform field and a plane wave field

The 4-potential

$$A_\mu^{comb} = -\frac{1}{2}F_{\mu\nu}x^\nu + a_\mu(nx), \quad (39)$$

where $a_\mu(\phi)$ is a vector-valued function of a real variable ϕ and n is a normalized isotropic vector $n^\mu = (1, \mathbf{n})$,

$$n^2 = 0, \qquad \mathbf{n}^2 = 1, \quad (40)$$

produces the field

$$F_{\mu\nu}^{comb}(nx) = F_{\mu\nu} + f_{\mu\nu}(nx), \tag{41}$$

which is a superposition of the constant field $F_{\mu\nu}$ and the plane-wave field

$$f_{\mu\nu}(nx) = n_\mu a'_\nu(nx) - n_\nu a'_\mu(nx)$$

Without loss of generality we may choose a_μ to be transversal,

$$n^\mu a_\mu(\phi) = 0. \tag{42}$$

The dependence of SF $\Phi[x, e_0]$ on the trajectory $x^\mu(\tau)$ is twofold. In addition to the direct dependence (see eq.(18)), there is an indirect one through the external field. In the case under consideration the field depends on $x_\mu(\tau)$ only through the scalar combination $nx(\tau)$. Replacing the latter by an auxiliary scalar trajectory $\phi(\tau)$ one obtains

$$\begin{aligned}
\tilde{\Phi}[x, \phi, e_0] = & \left[m + (2e_0)^{-1}\dot{x}^\mu \star \tilde{K}_{\mu\lambda} \left(2\eta^{\lambda\kappa} - ge_0\tilde{B}^{\lambda\kappa} \right) \gamma_\kappa \right. \\
& - \frac{ig}{4} \left(me_0 + \dot{x}^\mu \star \tilde{K}_{\mu\lambda}\gamma^\lambda \right) \tilde{B}_{\kappa\nu}\sigma^{\kappa\nu} + m\frac{g^2e_0^2}{16}\tilde{B}_{\alpha\beta}\tilde{B}^{*\alpha\beta}\gamma^5 \right] \\
& \times \exp \left\{ -\frac{e_0}{2} \int_0^g dg' \mathrm{Tr}\tilde{\mathcal{R}}(g') \star \mathcal{F}^{comb}(\phi) \right\},
\end{aligned} \tag{43}$$

where $\mathcal{F}_{\mu\nu}^{comb}(\phi|\tau - \tau') = [F_{\mu\nu} + f_{\mu\nu}(\phi(\tau))]\delta(\tau, \tau')$ and

$$\tilde{B}_{\mu\nu} = F_{\mu\lambda}^{comb}(\phi) \star \tilde{K}^\lambda{}_\nu, \quad \tilde{K}_{\mu\nu} = \eta_{\mu\nu} + ge_0\tilde{\mathcal{R}}_{\mu\nu}(g) \star F^{comb}(\phi), \tag{44}$$

$$\left[\eta\frac{\partial}{\partial\tau} - ge_0F^{comb}(\phi(\tau)) \right] \tilde{\mathcal{R}}(q|\tau, \tau') = \eta\delta(\tau, \tau'), \tag{45}$$

$$\tilde{\mathcal{R}}(g|1, \tau) = -\tilde{\mathcal{R}}(g|0, \tau), \quad \forall\tau \in (0, 1). \tag{46}$$

Obviously,

$$\tilde{\mathcal{R}}(g)\Big|_{\phi(\tau)=nx(\tau)} = \mathcal{R}(g), \quad \tilde{K}\Big|_{\phi(\tau)=nx(\tau)} = K, \quad \tilde{B}\Big|_{\phi(\tau)=nx(\tau)} = B, \tag{47}$$

and, therefore,

$$\tilde{\Phi}[x, nx, e_0] = \Phi[x, e_0]. \tag{48}$$

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Inserting the integral of a δ-function,

$$\int D\phi \, D\lambda \, e^{i\lambda \star (\phi - nx)} = 1,$$

into the RHS of eq.(16) and using (48) one transforms the path integral (16) into a quasi-Gaussian one of simple form

$$
\begin{aligned}
S^c(x_{out}, x_{in}) = \frac{i}{2} \int_0^\infty de_0 \int D\phi \, D\lambda \, e^{i\lambda \star \phi} \int_{x_{in}}^{x_{out}} Dx \, M(e_0) \tilde{\Phi}[x, \phi, e_0] \\
\times \exp\{i\tilde{I}[x, \phi, e_0] - i\lambda \star (nx)\}.
\end{aligned}
\tag{49}
$$

The action functional

$$\tilde{I}[x, \phi, e_0] = -\frac{1}{2e_0}\dot{x} \star \dot{x} - \frac{e_0}{2}m^2 - \frac{g}{2}x \star \bar{\mathcal{F}} \star \dot{x} - ga(\phi) \star \dot{x} \tag{50}$$

(where $\bar{\mathcal{F}}(\tau, \tau') = F\delta(\tau - \tau')$) contains only linear and bilinear terms in x (and the bilinear part does not depend on the wave potential a_μ). SF $\tilde{\Phi}[x, \phi, e_0]$ is linear in x and (following the same way of reasoning as in the case of a constant field) one finds

$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF}\right)^{-\frac{1}{2}} \int D\phi \, D\lambda \, e^{i\lambda \star (\phi - nx_q)} \tilde{\Phi}[x_q, \phi, e_0] e^{i\tilde{I}[x_q, \phi, e_0]}, \tag{51}$$

where x_q is the solution to the equation

$$\ddot{x}_q - ge_0 F \dot{x}_q = e_0 \lambda n - e_0 g a'(\phi) \dot{\phi}, \tag{52}$$

obeying the boundary conditions

$$x_q(0) = x_{in}, \qquad x_q(1) = x_{out}. \tag{53}$$

Introducing an appropriate Green function $\mathcal{G} = \mathcal{G}(\tau, \tau')$ for the second-order operator,

$$\left[\eta \frac{\partial^2}{\partial \tau^2} - ge_0 F \frac{\partial}{\partial \tau}\right] \mathcal{G}(\tau, \tau') = \eta \delta(\tau - \tau'), \tag{54}$$

$$\mathcal{G}(0, \tau) = \mathcal{G}(1, \tau) = \mathcal{G}(\tau, 0) = \mathcal{G}(\tau, 1) = 0, \quad \forall \tau \in (0, 1), \tag{55}$$

one presents this solution in the form

$$x_q = x_{cl} + e_0 \mathcal{G} \star \left(\lambda n - ga'(\phi)\dot{\phi}\right). \tag{56}$$

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The value of the action functional $\tilde{I}[x, \phi, e_0]$ on the solution x_q is given by

$$\tilde{I}[x_q, \phi, e_0] = \bar{I}[x_{cl}, e_0] - ga(\phi) \star \dot{x}_{cl}$$
$$- \frac{e_0}{2} \left(ga'(\phi)\dot{\phi} - \lambda n \right) \star \mathcal{G} \star \left(ga'(\phi)\dot{\phi} - \lambda n \right) + \lambda n \star (x_q - x_{cl}), \qquad (57)$$

where

$$\bar{I}[x, e_0] = -\frac{1}{2e_0}\dot{x} \star \dot{x} - \frac{e_0}{2}m^2 - \frac{g}{2}x \star \bar{\mathcal{F}} \star \dot{x} \qquad (58)$$

is the action in a uniform constant field F.

The functional integral over λ in (49) is a quasi-Gaussian one of simple form (let us remind that x_q is linear in λ, see eq.(56)) and the integration can be done explicitly. The result is a formula for the propagator in which the only functional integration is over the scalar trajectory $\phi(\tau)$. However, the latter integration is hardly to be performed explicitly in the general case (for arbitrary $a_\mu(\phi)$). Nevertheless, there exists a specific combination [5] for which the integration can be done and explicit formulae for the propagator to be derived. The latter are comparable with the corresponding Schwinger-type formulae [10], which are also explicit in this case.

Namely, let us choose the wave vector n to coincide with a real eigenvector of the matrix F (see Appendix A),

$$F_{\mu\nu}n^\nu = -\mathcal{E}n_\mu, \qquad n^2 = 0, \qquad \mathbf{n}^2 = 1, \qquad (59)$$

In this case $nx_q = nx_{cl}$, and, moreover, the action functional (50) is "on-shell" invariant with respect to longitudinal shifts

$$x_q(\tau) \rightarrow x_q(\tau) + \alpha(\tau)n, \qquad \alpha(0) = \alpha(1) = 0, \qquad (60)$$

by virtue of (59) and the transversality (42) of the wave potential a_μ. Then $\tilde{I}[x_q, \phi, e_0]$ does not depend on λ,

$$\tilde{I}[x_q, \phi, e_0] = \tilde{I}[x_{tr}, \phi, e_0] \qquad (61)$$

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where

$$x_{tr} = x_{cl} - ge_0 \mathcal{G} \star \left(a'(\phi)\dot{\phi} \right) \tag{62}$$

is a solution to the equation

$$\ddot{x}_{tr} - ge_0 F \dot{x}_{tr} = -ge_0 a'(\phi)\dot{\phi}, \tag{63}$$

obeying the boundary conditions (52). However SF $\tilde{\Phi}[x_q, \phi, e_0]$ does not show this invariance and, therefore, is λ-dependent. Presenting x_q as a sum $x_q = x_{tr} + e_0 \mathcal{G} n \star \lambda$, and substituting it into the expansion of SF in the antisymmetrized products of γ-matrices,

$$\tilde{\Phi}[x, \phi, e_0] = \left[e_0^{-1} \dot{x}^\mu \star \tilde{K}_{\mu\nu} \left(\gamma^\nu + \frac{ge_0}{4} \tilde{B}_{\alpha\beta} \gamma^{[\nu} \gamma^\alpha \gamma^{\beta]} \right) \right.$$
$$\left. +m \left(1 + \frac{ge_0}{4} \tilde{B}_{\alpha\beta} \gamma^{[\alpha} \gamma^{\beta]} + \frac{g^2 e_0^2}{32} \tilde{B}_{\alpha\beta} \tilde{B}_{\mu\nu} \gamma^{[\alpha} \gamma^\beta \gamma^\mu \gamma^{\nu]} \right) \right] \Lambda[\phi, e_0], \tag{64}$$

$$\Lambda[\phi, e_0] = \exp \left\{ -\frac{e_0}{2} \int_0^g dg' \mathrm{Tr} \tilde{\mathcal{R}}(g') \star \mathcal{F}^{comb}(\phi) \right\}. \tag{65}$$

we obtain,

$$\tilde{\Phi}[x_q, \phi, e_0] = \tilde{\Phi}[x_{tr}, \phi, e_0] + \lambda \star l[\phi, e_0], \tag{66}$$

$$l[\phi, e_0] = n^\kappa \mathcal{G}_{\kappa\mu}^{(r)} \star \tilde{K}^\mu{}_\nu \left(\gamma^\nu + \frac{ge_0}{4} \tilde{B}_{\alpha\beta} \gamma^{[\nu} \gamma^\alpha \gamma^{\beta]} \right) \Lambda[\phi, e_0], \tag{67}$$

$$\mathcal{G}^{(r)}(\tau, \tau') = \frac{\partial}{\partial \tau'} \mathcal{G}(\tau, \tau'). \tag{68}$$

It turns out that $l[\phi, e_0]$ does not depend on ϕ. First, expanding $\tilde{\mathcal{R}}$ in powers of f and using (59) and (42) one derives that $\Lambda[\phi, e_0]$ coincides with the expression

$$\bar{\Lambda}(e_0) = \exp \left\{ \frac{e_0}{2} \int_O^g dg' \mathrm{Tr} \bar{\mathcal{R}}(g) \star \bar{\mathcal{F}} \right\}. \tag{69}$$

Second[1],

$$n_\mu \tilde{K}^\mu{}_\nu = n_\mu \bar{K}^\mu{}_\nu = \frac{1}{2} \left(n \bar{K} \bar{n} \right) n_\nu. \tag{70}$$

---

[1]In this section we denote by $\bar{\mathcal{R}}(g)$, $\bar{K}$, $\bar{B}$ the quantities given by (19), i.e., corresponding to the case of a constant uniform field F.

Indeed, using the definitions (47) one finds that $\tilde{K}$ satisfies the equation

$$\left[\frac{\partial}{\partial \tau} - ge_0(F + f(\phi(t)))\right] \tilde{K}(\tau) = 0, \tag{71}$$

and the boundary conditions

$$\tilde{K}(0) + \tilde{K}(1) = 2\eta. \tag{72}$$

Multiplying (71) by n and using the properties (59), (42) we find

$$\left(\frac{\partial}{\partial \tau} - ge_0\mathcal{E}\right) n\tilde{K} = 0, \qquad n\tilde{K}(0) + n\tilde{K}(1) = 2n. \tag{73}$$

At the same time $n\bar{K}$ obeys (73). Therefore, $n\tilde{K}$ and $n\bar{K}$ coincide. Then using (42) and the properties of $n, \bar{n}$ (see Appendix A) one gets (70). Third, using the same properties of the electromagnetic field one can derive

$$\tilde{B}_{\alpha\beta} = \bar{B}_{\alpha\beta} + n_\alpha b_\beta - b_\alpha n_\beta, \tag{74}$$

where b_α depends on ϕ and $\bar{B}$ is given by (19). Substituting (74) into (67) and using (59) one finds

$$l[\phi, e_0] =$$
$$\frac{1}{4}\left(n\mathcal{G}^{(r)}\bar{n}\right) \star \left(n\bar{K}\bar{n}\right)\left[n_\nu\gamma^\nu + \frac{ge_0}{4}n_\nu\left(\bar{B}_{\alpha\beta} + n_\alpha b_\beta - b_\alpha n_\beta\right)\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right]\bar{\Lambda}(e_0).$$

and the contribution of the ϕ-dependent terms vanishes by virtue of the complete antisymmetry of $\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}$. Therefore $l[\phi, e_0]$ can be replaced by

$$\bar{l}(e_0) = n^\kappa \mathcal{G}^{(r)}_{\kappa\mu} \star \bar{K}^\mu{}_\nu \left(\gamma^\nu + \frac{ge_0}{4}\bar{B}_{\alpha\beta}\gamma^{[\nu}\gamma^\alpha\gamma^{\beta]}\right)\bar{\Lambda}(e_0). \tag{75}$$

Substituting (75) into (66) and then into (51), using (61) and

$$\lambda(\tau)e^{i\lambda\star\phi} = -i\frac{\delta}{\delta\phi(\tau)}e^{i\lambda\star\phi},$$

and integrating by parts we find

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$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF} \right)^{-\frac{1}{2}} \int D\phi\, D\lambda\, e^{i\lambda \star (\phi - nx_{cl})}$$
$$\times \left[\tilde{\Phi}[x_{tr}, \phi, e_0] - \left(\frac{\delta}{\delta\phi} \tilde{I}[x_{tr}, \phi, e_0] \right) \star \bar{l}(e_0) \right] \exp \left\{ i\tilde{I}[x_{tr}, \phi, e_0] \right\}. \tag{76}$$

Inserting the derivative,

$$\frac{\delta}{\delta\phi} \tilde{I}[x_{tr}, \phi, e_0] = -a'(\phi)\dot{x}_{tr},$$

and using (74), (67) we transform (76) to the following form

$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF} \right)^{-\frac{1}{2}} \int D\phi\, D\lambda\, e^{i\lambda \star (\phi - nx_{cl})}$$
$$\times \tilde{\Phi}[\tilde{x}, \phi, e_0] \exp \left\{ i\tilde{I}[x_{tr}, \phi, e_0] \right\}, \tag{77}$$

where

$$\tilde{x}^\mu = x_{tr}^\mu + ge_0 n^\mu a'_\kappa(\phi)\dot{x}_{tr}^\kappa.$$

One can straightforwardly check that $\tilde{x}$ satisfies the equation

$$\ddot{\tilde{x}}_\mu - ge_0 \left(F_{\mu\nu} + n_\mu a'_\nu(\phi) \right) \dot{\tilde{x}}^\nu = -ge_0 a'_\mu(\phi)\dot{\phi} \tag{78}$$

and the boundary conditions (52). The trajectory x_{tr} in the action $\tilde{I}$ can be replaced by $\tilde{x}$ due to the invariance of the action $\tilde{I}[x, \phi, e_0]$ under the longitudinal shifts (60). The integration over λ and ϕ is straightforward now. One needs only to take into account that $\tilde{x}|_{\phi=nx_{cl}} \equiv x_{comb}$ is the solution (subjected to the boundary conditions (52)) to the equation of motion

$$\ddot{x}^\mu_{comb} - ge_0 \left(F + f(nx_{comb}) \right)^\mu{}_\nu \dot{x}^\nu_{comb} = 0. \tag{79}$$

Indeed, eq.(78) turns out to be equivalent to eq.(79) when $\phi = nx_{cl}$ and the relation $nx_{comb} = nx_{cl}$ is taken into account. Therefore,

$$\tilde{\Phi}[\tilde{x}, \phi, e_0]\Big|_{\phi=nx_{cl}} = \Phi[x_{comb}, e_0], \qquad \tilde{I}[\tilde{x}, \phi, e_0]\Big|_{\phi=nx_{cl}} = I[x_{comb}, e_0]. \tag{80}$$

Finally, we get

$$S^c = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF} \right)^{-\frac{1}{2}} \Phi[x_{comb}, e_0] e^{I[x_{comb}, e_0]}, \tag{81}$$

where

$$\Phi[x_{comb}, e_0] = \left[e_0^{-1} \dot{x}^\mu_{comb} \star K_{\mu\nu} \left(\gamma^\nu + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\nu} \gamma^\alpha \gamma^{\beta]} \right) \right.$$
$$\left. + m \left(1 + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\alpha} \gamma^{\beta]} + \frac{g^2 e_0^2}{32} B_{\alpha\beta} B_{\mu\nu} \gamma^{[\alpha} \gamma^\beta \gamma^\mu \gamma^{\nu]} \right) \right] \bar{\Lambda}(e_0). \tag{82}$$

The vector $\dot{x}_{comb}$ satisfies (79) and can be presented as

$$\dot{x}_{comb}(\tau) = T_A \exp \left\{ -ge_0 \int_\tau^1 F^{comb}(nx_{cl}(\tau)) d\tau \right\} \dot{x}(1), \tag{83}$$

where T_A denotes the antichronological product. On the other hand the tensor trajectory $K(\tau)$ satisfies (71) where one has to replace ϕ by nx_{cl}. Therefore

$$K(\tau) = T_A \exp \left\{ -ge_0 \int_\tau^1 F^{comb}(nx_{cl}) d\tau \right\} K(1), \tag{84}$$

and[2]

$$\dot{x}_{comb} \star K = \dot{x}_{comb}(1) K(1). \tag{85}$$

Substituting (85) into (82) and taking into account the relation $B_{[\alpha\beta} B_{\mu\nu]} = \bar{B}_{[\alpha\beta} \bar{B}_{\mu\nu]}$ we find

$$\Phi[x_{comb}, e_0] = \left[e_0^{-1} \dot{x}^\mu_{comb}(1) K_{\mu\nu}(1) \left(\gamma^\nu + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\nu} \gamma^\alpha \gamma^{\beta]} \right) \right.$$
$$\left. + m \left(1 + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\alpha} \gamma^{\beta]} + \frac{g^2 e_0^2}{32} \bar{B}_{\alpha\beta} \bar{B}_{\mu\nu} \gamma^{[\alpha} \gamma^\beta \gamma^\mu \gamma^{\nu]} \right) \right] \bar{\Lambda}(e_0). \tag{86}$$

A representation for the propagator in this specific field combination (characterized by the relation (59)) was given [12,10] in terms of proper-time integral only. Another more complicated representation has been obtained before in [5]. In our notation the representation [12,10] can be written as

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[2]The operator $T_A \exp \left\{ -ge_0 \int_\tau^1 F^{comb}(nx_{cl}) d\tau \right\}$ preserves the scalar product due to the antisymmetry of the stress tensor.

$$S^c(x_{out}, x_{in}) = \left[\gamma^\mu \left(i\frac{\partial}{\partial x_{out}^\mu} - gA_\mu^{comb}(nx_{out})\right) + m\right]$$

$$\times \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF}\right)^{-\frac{1}{2}} e^{iI[x_{comb}, e_0]} \Delta[nx_{cl}, e_0], \tag{87}$$

where

$$\Delta[nx_{cl}, e_0] = T \exp\left\{-\frac{ige_0}{4} \int_0^1 d\tau \, (F + f(nx_{cl}(\tau)))_{\mu\nu} \sigma^{\mu\nu}\right\} \tag{88}$$

$$= \exp\left\{-\frac{ige_0}{4} F_{\mu\nu}\sigma^{\mu\nu}\right\} - \frac{ige_0}{4} \int_0^1 d\tau \left(e^{\frac{ge_0}{2}F(1-2\tau)} f(nx_{cl}(\tau)) e^{-\frac{ge_0}{2}F(1-2\tau)}\right)_{\mu\nu} \sigma^{\mu\nu}, \tag{89}$$

and T denotes chronological product. Taking the derivative in (87) one can use the relation

$$\frac{\partial}{\partial x_{out}^\mu} I[x_{comb}, e_0] = -p_\mu(1), \tag{90}$$

where

$$p_\mu(\tau) = -\left.\frac{\delta}{\delta \dot{x}^\mu} I[x, e_0]\right|_{x=x_{comb}},$$

is the on-shell momentum, in particular,

$$p(1) = e_0^{-1} \dot{x}_{comb}(1) + ge_0 A^{comb}(nx_{out}). \tag{91}$$

On the other hand,

$$\gamma^\mu \frac{\partial}{\partial x_{out}^\mu} \Delta[nx_{cl}, e_0] = 0. \tag{92}$$

Indeed, one gets from eq.(32, with the aid of eq.(59),

$$\gamma^\mu \frac{\partial}{\partial x_{out}^\mu} (nx_{cl}(\tau)) = \frac{e^{ge_0 \mathcal{E}\tau} - 1}{e^{ge_0 \mathcal{E}} - 1} n_\mu \gamma^\mu. \tag{93}$$

Then, using the representation (89) for $\Delta[nx_{cl}, e_0]$, eqs.(40) and (42), and the properties of the γ-matrices, one easily derives (92). Differentiating in (87), one obtains, with the aid of (89), (90) and (92),

$$S^c(x_{out}, x_{in}) = \frac{1}{32\pi^2} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF}\right)^{-\frac{1}{2}} \Psi_S^{comb}(x_{out}, x_{in}, e_0) e^{iI[x_{comb}, e_0]}, \tag{94}$$

where

$$\Psi_S^{comb}(x_{out}, x_{in}, e_0) = \left(e_0^{-1}\dot{x}_{comb}^{\mu}(1)\gamma_{\mu} + m\right)\Delta[nx_{cl}, e_0].$$

Using the identities (B12) and (B13) one can verify that

$$\Psi_S^{comb}(x_{out}, x_{in}, e_0) = \Phi[x_{comb}, e_0].$$

Thus the representations (81) and (87) are equivalent.

III. SPIN FACTOR IN $2+1$ DIMENSIONS

A. Derivation of spin factor

In $2+1$ dimensions the equation for the Dirac propagator has the form

$$[\gamma^{\mu}(i\partial_{\mu} - gA_{\mu}(x)) - m]S^c(x, y) = -\delta^3(x-y), \tag{95}$$

where γ matrices in $2+1$ dimensions can be taken, for example, in the form $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^2$, $\gamma^2 = -i\sigma^1$, $[\gamma^{\mu}, \gamma^{\nu}]_+ = 2\eta^{\mu\nu}$, $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1)$, $\mu, \nu = 0, 1, 2$. In this particular case they obey the relations

$$[\gamma^{\mu}, \gamma^{\nu}] = -2i\epsilon^{\mu\nu\lambda}\gamma_{\lambda}, \qquad \gamma^{\mu} = \frac{i}{2}\epsilon^{\mu\nu\lambda}\gamma_{\nu}\gamma_{\lambda}. \tag{96}$$

In the paper [13] a path integral representation for the Dirac propagator was obtained in arbitrary odd dimensions. In particular, in the case under consideration this representation reads

$$S^c = \frac{1}{2}\exp\left(i\gamma^{\mu}\frac{\partial_{\ell}}{\partial\theta^{\mu}}\right)\int_0^{\infty}de_0 \int d\chi_0 \int_{e_0} M(e)De \int_{\chi_0} D\chi \int_{x_{in}}^{x_{out}} Dx \int D\pi \int D\nu \tag{97}$$

$$\times \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{i\int_0^1\left[-\frac{\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}_{\mu}A^{\mu} + iegF_{\mu\nu}\psi^{\mu}\psi^{\nu}\right.\right.$$

$$\left.\left. +\chi\left(\frac{2i}{e}\epsilon_{\mu\nu\lambda}\dot{x}^{\mu}\psi^{\nu}\psi^{\lambda} - m\right) - i\psi_{\mu}\dot{\psi}^{\nu} + \pi\dot{e} + \nu\dot{\chi}\right]d\tau + \psi_{\mu}(1)\psi^{\mu}(0)\right\}\Big|_{\theta=0},$$

where $x(\tau)$, $p(\tau)$, $e(\tau)$, $\pi(\tau)$ are even and $\psi(\tau)$, $\chi_1(\tau), \chi_2(\tau)$, $\nu_1(\tau), \nu_2(\tau)$ are odd trajectories, obeying the boundary conditions $x(0) = x_{in}$, $x(1) = x_{out}$, $e(0) = e_0$, $\chi(0) = \chi_0$, $\psi(0) + \psi(1) = \theta$, and the notations used are

$$\chi = \chi_1\chi_2, \ \nu\dot\chi = \nu_1\dot\chi_1 + \nu_2\dot\chi_2, \ d\chi = d\chi_1\,d\chi_2, \ D\chi = D\chi_1\,D\chi_2, \ D\nu = D\nu_1\,D\nu_2\,.$$

The measure $M(e)$ is defined by the eq. (3) in the corresponding dimensions[3], and

$$\mathcal{D}\psi = D\psi \left[\int_{\psi(0)+\psi(1)=0} D\psi \exp\left\{ \int_0^1 \psi_\mu \dot\psi^\mu d\tau \right\} \right]^{-1}.$$

Integrating over the Grassmann variables in the same way as in the case of $3+1$ dimensions we get

$$S^c(x_{out}, x_{in}) = \frac{i}{2} \int_0^\infty de_0 M(e_0) \int_{x_{in}}^{x_{out}} Dx\, \Phi[x, e_0] \exp\left\{ iI[x, e_0] \right\}, \tag{98}$$

where

$$\Phi[x, e_0] = \left[\left(m + \frac{i}{e_0} \int_0^1 d\tau\, \epsilon_{\mu\nu\lambda} \dot{x}^\mu(\tau) \mathcal{R}^{\nu\lambda}(g|\tau, \tau) \right) \left(1 + \frac{ge_0}{4} B_{\alpha\beta} \gamma^\alpha \gamma^\beta \right) \right]$$
$$+ \frac{i}{2e_0} \int_0^1 d\tau\, \epsilon_{\mu\nu\lambda} \dot{x}^\mu(\tau) K^\nu{}_\alpha(\tau) K^\lambda{}_\beta(\tau) \gamma^\alpha \gamma^\beta \right] \exp\left\{ -\frac{e_0}{2} \int_0^g dg'\mathrm{Tr}\mathcal{R}(g) \star \mathcal{F} \right\} \tag{99}$$

is SF and $I[x, e_0]$, $\mathcal{R}(g) \equiv \mathcal{R}(g|\tau, \tau')$, $K \equiv K(\tau)$, B, $\mathcal{F}$ are defined by (17), (14), and (8) respectively.

Due to the relations (96) one can also present SF in the form

$$\Phi[x, e_0] = \left\{ m + \frac{i}{e_0} \dot{x} \star r(g) + \left[\left(-i\frac{ge_0}{4} m + \frac{g}{4} \dot{x} \star r(g) \right) u_\alpha \right.\right.$$
$$\left.\left. + \frac{1}{2e_0} \left(\dot{x} \star T \right)_\alpha \right] \gamma^\alpha \right\} \exp\left\{ -\frac{e_0}{2} \int_0^g dg'\mathrm{Tr}\mathcal{R}(g') \star \mathcal{F} \right\}, \tag{100}$$

where

$$r_\mu(g) \equiv r_\mu(g|\tau) = \epsilon_{\mu\nu\lambda}\mathcal{R}^{\nu\lambda}(g|\tau, \tau), \qquad u^\mu = \epsilon^{\mu\alpha\beta} B_{\alpha\beta}, \qquad T_\mu{}^\rho = \epsilon_{\mu\nu\lambda}\epsilon^{\rho\alpha\beta} K^\nu{}_\alpha K^\lambda{}_\beta.$$

---

[3]We will refer to some formulae from the previous sections without specifying that the number of dimensions is $2+1$ now.

B. Dirac propagator in constant uniform field in $2+1$ dimensions

In the case of constant uniform field $F_{\mu\nu} = const$ one can calculate the propagator explicitly integrating over the bosonic trajectories. Following the same way as in Sect.II and taking into account that in $2+1$ dimensions

$$\frac{i}{2}M(e_0) \int_0^0 Dx \, \exp\left\{-\frac{i}{2e_0}\dot{x} \star \dot{x}\right\} = \frac{e^{i\frac{\pi}{4}}}{2(2\pi e_0)^{3/2}}$$

one gets for the propagator (98)

$$S^c = \frac{e^{i\frac{\pi}{4}}}{2(4\pi)^{3/2}} \int_0^\infty de_0 \left(\det \frac{\sinh \frac{ge_0 F}{2}}{gF}\right)^{-\frac{1}{2}} e^{iI[x_{cl},e_0]} \Phi[x_{cl}, e_0], \tag{101}$$

where x_{cl}, $\mathcal{R}(g)$, K, B are given by (32), (19).

The antisymmetric matrices $F_{\mu\nu}$ can be classified by the value of the invariant φ (see Appendix A). In the case $\varphi^2 > 0$ one can find a Lorentz frame in which the magnetic field vanishes. On the other hand, $\varphi^2 < 0$ implies that the electric field vanishes in an appropriate Lorentz frame. The case $\varphi^2 = 0$, $F \neq 0$ corresponds to nonvanishing electric and magnetic fields of 'equal magnitude' (and this property is Lorentz invariant). We will consider the case $\varphi^2 \neq 0$. The case $\varphi^2 = 0$ can be easily treated, e.g. taking the limit $\varphi \to 0$.

Presenting $\Phi[x_{cl}, e_0]$ in the case under consideration one can avoid the explicit integrations over τ. Indeed, due to the specific form of $\mathcal{R}(g)$ the term containing it in (99) vanishes. On the other hand,

$$\left(\frac{d}{d\tau} - ge_0 F\right) K = 0, \qquad \left(\frac{\partial}{\partial\tau} - ge_0 F\right) \dot{x}_{cl} = 0$$

imply

$$\dot{x}_{cl}(\tau) = e^{ge_0 F(\tau-1)} \dot{x}_{cl}(1), \qquad K(\tau) = e^{ge_0 F(\tau-1)} K(1).$$

Taking into account that $e^{ge_0 F(\tau-1)}$ is an operator respecting the scalar product, one can easily perform the second integration over τ in (99). Finally, calculating the determinants involved by means of (A12), one gets

$$S^c = \sqrt{\frac{i}{16(2\pi)^3}} \int_0^\infty \frac{de_0}{\sqrt{e_0}} \frac{g\varphi}{\sinh \frac{ge_0\varphi}{2}} e^{iI[x_{cl},e_0]} \Phi[x_{cl},e_0],$$

$$\Phi[x_{cl},e_0] = \left[m\left(1 + \frac{ge_0}{4} B_{\alpha\beta}\gamma^\alpha\gamma^\beta\right) + \frac{i}{2e_0}\epsilon_{\mu\nu\lambda}\dot{x}^\mu(1) K^\nu{}_\alpha K^\lambda{}_\beta \gamma^\alpha\gamma^\beta \right] \cosh\frac{ge_0\varphi}{2}. \tag{102}$$

On the other hand one can obtain a representation for the propagator using Schwinger proper-time method (we do not present the calculations here). Such a representation has the form

$$S^c(x_{out}, x_{in}) = \left[\gamma^\mu \left(i\frac{\partial}{\partial x_{out}^\mu} + gA_\mu(x_{out}) \right) + m \right]$$

$$\times \sqrt{\frac{i}{16(2\pi)^3}} \int_0^\infty \frac{de_0}{\sqrt{e_0}} \frac{g\varphi}{\sinh\frac{ge_0\varphi}{2}} e^{iI[c_{cl},e_0]} e^{\frac{ge_0}{4} F_{\alpha\beta}\gamma^\alpha\gamma^\beta}. \tag{103}$$

To compare both representations we take the derivative in (103) and use

$$\frac{\partial}{\partial x_{out}^\mu} I[x_{cl},e_0] = -e_0^{-1} \left(\dot{x}_{cl}\right)_\mu (1) - gA_\mu(x_{out}).$$

Then one obtains

$$S^c(x_{out}, x_{in}) = \frac{e^{i\frac{\pi}{4}}}{4(2\pi)^{3/2}} \int_0^\infty \frac{de_0}{\sqrt{e_0}} \frac{g\varphi}{\sinh\frac{ge_0\varphi}{2}} \Psi_S(x_{out}, x_{in}, e_0) e^{iI[x_{cl},e_0]}, \tag{104}$$

where

$$\Psi_S(x_{out}, x_{in}, e_0) = \left(e_0^{-1}\gamma_\mu \dot{x}_{cl}^\mu(1) + m \right) e^{\frac{ge_0}{4} F_{\alpha\beta}\gamma^\alpha\gamma^\beta}. \tag{105}$$

Comparing (102) and (104) using the identities (see Appendix B)

$$\exp\left\{ \frac{ge_0}{4} F_{\alpha\beta}\gamma^\alpha\gamma^\beta \right\} = \left(1 + \frac{ge_0}{4} B_{\alpha\beta}\gamma^\alpha\gamma^\beta\right) \cosh\frac{ge_0\varphi}{2}, \tag{106}$$

$$\gamma_\mu \exp\left\{ \frac{ge_0}{4} F_{\alpha\beta}\gamma^\alpha\gamma^\beta \right\} = \frac{i}{2e_0}\epsilon_{\mu\nu\lambda}K^\nu{}_\alpha(1) K^\lambda{}_\beta(1)\gamma^\alpha\gamma^\beta \cosh\frac{ge_0\varphi}{2}. \tag{107}$$

one can verify that both the representations coincide.

APPENDIX A: SOME PROPERTIES OF ANTISYMMETRIC TENSORS

1. Antisymmetric tensors in $3+1$ dimensions

The antisymmetric matrix $F_{\mu\nu}$ formed by the components of the stress tensor has in the general case of nonvanishing invariants,

$$\mathbf{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad \mathbf{G} = -\frac{1}{4} F_{\mu\nu} F^{*\mu\nu}, \tag{A1}$$

four isotropic eigenvectors, namely,

$$F_{\mu\nu} n^\nu = -\mathcal{E} n_\mu, \quad F_{\mu\nu} \bar{n}^\nu = \mathcal{E} \bar{n}_\mu,$$

$$F_{\mu\nu} m^\nu = i\mathcal{H} m_\mu, \quad F_{\mu\nu} \bar{m}^\nu = -i\mathcal{H} \bar{m}_\mu, \tag{A2}$$

where

$$\mathcal{E} = \left[\left(\mathbf{F}^2 + \mathbf{G}^2 \right)^{1/2} - \mathbf{F} \right]^{1/2}, \qquad \mathcal{H} = \left[\left(\mathbf{F}^2 + \mathbf{G}^2 \right)^{1/2} + \mathbf{F} \right]^{1/2}. \tag{A3}$$

The eigenvectors are supposed to be normalized,

$$\bar{n}^\mu n_\mu = -\bar{m}^\mu m_\mu = 2, \tag{A4}$$

while all other scalar products vanish,

$$n^2 = \bar{n}^2 = m^2 = \bar{m}^2 = nm = n\bar{m} = \bar{n}m = \bar{n}\bar{m} = 0. \tag{A5}$$

Then the matrix F can be presented in the form

$$F_{\mu\nu} = \frac{\mathcal{E}}{2} \left(\bar{n}_\mu n_\nu - n_\mu \bar{n}_\nu \right) + \frac{i\mathcal{H}}{2} \left(\bar{m}_\mu m_\nu - m_\mu \bar{m}_\nu \right), \tag{A6}$$

and, therefore, F^2 has the spectral decomposition

$$F^2 = \mathcal{E}^2 P_{\mathcal{E}} + \mathcal{H}^2 P_{\mathcal{H}} \tag{A7}$$

where

$$\left(P_{\mathcal{E}} \right)_{\mu\nu} = \frac{1}{2} \left(\bar{n}_\mu n_\nu + n_\mu \bar{n}_\nu \right), \qquad \left(P_{\mathcal{H}} \right)_{\mu\nu} = \frac{1}{2} \left(\bar{m}_\mu m_\nu + m_\mu \bar{m}_\nu \right) \tag{A8}$$

are orthogonal projection operators onto some two-dimensional subspaces,

$$P_{\mathcal{E}}^2 = P_{\mathcal{E}}, \quad P_{\mathcal{H}}^2 = P_{\mathcal{H}}, \quad P_{\mathcal{E}} P_{\mathcal{H}} = P_{\mathcal{H}} P_{\mathcal{E}} = 0, \tag{A9}$$

$$P_{\mathcal{E}} + P_{\mathcal{H}} = \eta, \qquad \mathrm{tr} P_{\mathcal{E}} = \mathrm{tr} P_{\mathcal{H}} = 2. \tag{A10}$$

2. Antisymmetric tensors in $2+1$ dimensions

Let $F_{\mu\nu}$ be an antisymmetric matrix in $2+1$ dimensions. The antisymmetry implies $\mathrm{tr}F = 0$, $\det F = 0$, so that the sum and the product of the eigenvalues vanish. The eigenvalues are 0, φ, $-\varphi$, where the real number φ^2 coincides with the invariant

$$\varphi^2 = \frac{1}{2}\mathrm{tr}F^2.$$

In the case of nonvanishing φ there exist three eigenvectors of F, and F^2 is proportional to a projection operator P onto some two-dimensional subspace,

$$F^2 = \varphi^2 P, \qquad P^2 = P, \qquad \mathrm{tr}P = 2, \qquad PF = FP = F. \tag{A11}$$

Then, for an even function h,

$$h(F) = h(0)\,(1 - P) + h(\varphi)P, \tag{A12}$$

while for an odd one

$$h(F) = \frac{F}{\varphi}h(\varphi). \tag{A13}$$

The case of vanishing φ (and $F \neq 0$) corresponds to a nilpotent matrix, $F^3 = 0$.

APPENDIX B: SOME IDENTITIES INVOLVING γ-MATRICES

1. Gamma-matrix structure of Lorentz transformation in the spinor representation

Let us denote by $M(\omega)$ the expression in the RHS of eq.(38). We are going to check that the matrix-valued function $M(\lambda\omega)$ of a real parameter λ satisfies the differential equation

$$\frac{d}{d\lambda}M(\lambda\omega) = -\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}M(\lambda\omega) \tag{B1}$$

and the initial condition

$$M(0) = 1. \tag{B2}$$

The latter is trivial, so let us concentrate on the proof of the equation (B1).

With the derivatives

$$\frac{d}{d\lambda}\left(\det\cosh\frac{\lambda\omega}{2}\right)^{1/2} = \left(\det\cosh\frac{\lambda\omega}{2}\right)\text{tr}\left(\frac{\omega}{4}\tanh\frac{\lambda\omega}{2}\right),$$

$$\frac{d}{d\lambda}\tanh\frac{\lambda\omega}{2} = \frac{\omega}{2}\left(\eta - \tanh^2\frac{\lambda\omega}{2}\right),$$

one obtains

$$\frac{d}{d\lambda}M(\lambda\omega) = \left(\det\cosh\frac{\lambda\omega}{2}\right)^{1/2}\left\{\frac{1}{4}\text{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\right.$$

$$-\frac{i}{8}\left[\text{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\tanh\frac{\lambda\omega}{2} + 2\omega\left(\eta - \tanh^2\frac{\lambda\omega}{2}\right)\right]_{\mu\nu}\sigma^{\mu\nu}$$

$$+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\left(\tanh\frac{\lambda\omega}{2}\right)_{\alpha\beta}\left[\frac{1}{4}\text{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\tanh\frac{\lambda\omega}{2}\right.$$

$$\left.\left. + \omega\left(\eta - \tanh^2\frac{\lambda\omega}{2}\right)\right]_{\mu\nu}\gamma^5\right\}. \tag{B3}$$

Then using the identity

$$4\epsilon^{\alpha\beta\mu\nu}(ST)_{\alpha\beta}T_{\mu\nu} = \epsilon^{\alpha\beta\mu\nu}T_{\alpha\beta}T_{\mu\nu}\,\text{tr}S, \tag{B4}$$

which is valid for any two matrices[4] S and T one can put the derivative (B3) into the form

$$\frac{d}{d\lambda}M(\lambda\omega) = \left(\det\cosh\frac{\lambda\omega}{2}\right)^{1/2}\left\{\frac{1}{4}\text{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\right.$$

$$-\frac{i}{8}\left[\text{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\tanh\frac{\lambda\omega}{2} + 2\omega\left(\eta - \tanh^2\frac{\lambda\omega}{2}\right)\right]_{\mu\nu}\sigma^{\mu\nu}$$

$$\left. + \frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\omega_{\alpha\beta}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\gamma^5\right\}. \tag{B5}$$

A representation for the product of two σ-matrices,

$$\sigma^{\alpha\beta}\sigma^{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}$$

$$-i\left(\eta^{\alpha\mu}\sigma^{\beta\nu} + \eta^{\beta\nu}\sigma^{\alpha\mu} - \eta^{\alpha\nu}\sigma^{\beta\mu} - \eta^{\beta\mu}\sigma^{\alpha\nu}\right) - \epsilon^{\alpha\beta\mu\nu}\gamma^5, \tag{B6}$$

_____

[4]We remind that, in accordance with our notation, $\text{tr}S = S^\alpha{}_\alpha = \eta^{\alpha\beta}S_{\beta\alpha}$, etc.

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is easily derived from the identities

$$\frac{1}{2}\{\sigma^{\alpha\beta}, \sigma^{\mu\nu}\} = \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu} - \epsilon^{\alpha\beta\mu\nu}\gamma^5, \tag{B7}$$

$$\frac{i}{2}[\sigma^{\alpha\beta}, \sigma^{\mu\nu}] = \eta^{\alpha\mu}\sigma^{\beta\nu} + \eta^{\beta\nu}\sigma^{\alpha\mu} - \eta^{\alpha\nu}\sigma^{\beta\mu} - \eta^{\beta\mu}\sigma_{\alpha\nu}. \tag{B8}$$

Using (B6), the well-known identity[5]

$$\sigma^{\kappa\rho}\gamma^5 = \frac{1}{2}\epsilon^{\kappa\rho\tau\sigma}\sigma_{\tau\sigma}, \tag{B9}$$

and the antisymmetry property $\omega_{\mu\nu} = -\omega_{\nu\mu}$, one finds

$$
\begin{aligned}
-\frac{i}{4}\omega\sigma^{\mu\nu}M(\lambda\omega) = &\left(\det\cosh\frac{\lambda\omega}{2}\right)^{1/2}\left[\frac{1}{4}\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\right.\\
&-\frac{i}{64}\epsilon^{\kappa\rho\mu\nu}\left(\tanh\frac{\lambda\omega}{2}\right)_{\kappa\rho}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\epsilon_{\alpha\beta\lambda\tau}\omega^{\alpha\beta}\sigma^{\lambda\tau}\\
&\left.-\frac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta} + \frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\omega_{\alpha\beta}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\gamma^5\right].
\end{aligned}
\tag{B10}
$$

Then, using the identity

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4} = -\sum_P(-1)^{[P]}\delta^{\beta_1}_{P(\alpha_1)}\delta^{\beta_2}_{P(\alpha_2)}\delta^{\beta_3}_{P(\alpha_3)}\delta^{\beta_4}_{P(\alpha_4)},$$

we get

$$
\begin{aligned}
-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}M(\lambda\omega) = &\left(\det\cosh\frac{\lambda\omega}{2}\right)^{1/2}\left[\frac{1}{4}\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\right.\\
&-\frac{i}{8}\left[\mathrm{tr}\left(\omega\tanh\frac{\lambda\omega}{2}\right)\tanh\frac{\lambda\omega}{2} + 2\omega\left(\eta - \tanh^2\frac{\lambda\omega}{2}\right)\right]_{\mu\nu}\sigma^{\mu\nu}\\
&\left.+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\omega_{\alpha\beta}\left(\tanh\frac{\lambda\omega}{2}\right)_{\mu\nu}\gamma^5\right\}.
\end{aligned}
\tag{B11}
$$

The RHS's of (B5) and (B11) coincide. Therefore $M(\lambda\omega)$ obeys eq.(B1). This completes the proof of formula (38).

---

[5]Let us remind that in our notation $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$.

2. Decompositions of some functions on γ-matrices

Let us consider the T-exponent (88) where F is a uniform constant field, f is a plane-wave field and (57) takes place. We are going to prove the identities

$$\Delta[nx_{cl}, e_0] = \left(1 + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\alpha} \gamma^{\beta]} + \frac{g^2 e_0^2}{16} \bar{B}_{\alpha\beta} \bar{B}^{*\alpha\beta} \gamma^5 \right) \bar{\Lambda}(e_0), \tag{B12}$$

$$\gamma^\mu \Delta[nx_{cl}, e_0] = K^\mu{}_\nu(1) \left(\gamma^\nu + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\nu} \gamma^\alpha \gamma^{\beta]} \right) \bar{\Lambda}(e_0), \tag{B13}$$

where B and K are defined by (14) for the combination as it was described while $\bar{B}$, corresponding to the case of constant uniform field is given by (19), and

$$\bar{\Lambda}(e_0) = \exp\left\{-\frac{e_0}{2} \int_0^g dg' \text{Tr} \bar{\mathcal{R}}(g') \star \bar{\mathcal{F}}\right\} = \cosh \frac{ge_0 \mathcal{E}}{2} \cos \frac{ge_0 \mathcal{H}}{2}.$$

Presenting the T-exponent in the form (89) and using eq.(38) one obtains

$$\Delta[nx_{cl}, e_0] = \left(1 + \frac{ge_0}{4} B_{\alpha\beta} \gamma^{[\alpha} \gamma^{\beta]} + \frac{g^2 e_0^2}{16} \bar{B}_{\alpha\beta} \bar{B}^{*\alpha\beta} \gamma^5 \right) \bar{\Lambda}(e_0) + Q_{\alpha\beta} \gamma^{[\alpha} \gamma^{\beta]} \tag{B14}$$

where

$$Q = \frac{ge_0}{4} \bar{\Lambda}(e_0) \left(\bar{B} - B\right) + \frac{1}{4} C, \tag{B15}$$

$$C = ge_0 \int_0^1 d\tau \, e^{\frac{ge_0}{2} F(1-2\tau)} f(nx_{cl}(\tau)) e^{-\frac{ge_0}{2} F(1-2\tau)}. \tag{B16}$$

In order to find a convenient representation for B we present K, which is a solution, obeying (52), to eq.(71, for $\phi = nx_{cl}$) in the form

$$K(\tau) = 2V(\tau) \left(\eta + V(1)\right)^{-1}, \tag{B17}$$

where

$$V(\tau) = T \exp\left\{ge_0 \int_0^\tau F^{comb}(nx_{cl}(\tau')) d\tau'\right\}. \tag{B18}$$

is the solution, subjected to (52), to the equation

$$\left[\frac{\partial}{\partial \tau} - ge_0 F - ge_0 f(nx_{cl}(\tau))\right] V(\tau) = 0. \tag{B19}$$

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Then, using the defining equation (14) for B (in which F must be understood as F^{comb}) and eqs.(B17), (B18), one derives

$$B = \frac{2}{ge_0} \left[\eta - 2 \left(\eta + V(1) \right)^{-1} \right]. \tag{B20}$$

Correspondingly, from eq.(19) we obtain

$$\bar{B} = \frac{2}{ge_0} \left[\eta - 2 \left(\eta + V_0(1) \right)^{-1} \right], \qquad V_0(\tau) = e^{ge_0 F \tau}. \tag{B21}$$

Solving eq.(B19) we find

$$V(1) = V_0^{1/2}(1) \left(\eta + C + \frac{C^2}{2} \right) V_0^{1/2}(1), \tag{B22}$$

by virtue of the nilpotency [10] of C. Then we substitute (B22) into (B20) and after straightforward transformations obtain

$$B - \bar{B} = \frac{1}{ge_0} \left(\cosh \frac{ge_0 F}{2} \right)^{-1} C \left(\cosh \frac{ge_0 F}{2} \right)^{-1}. \tag{B23}$$

One can verify, using the transversality (42) of a_μ, that[6]

$$C = P_{\mathcal{E}} C P_{\mathcal{H}} + P_{\mathcal{H}} C P_{\mathcal{E}}. \tag{B24}$$

On the other hand, due to the evenness of the function,

$$\left(\cosh \frac{ge_0 F}{2} \right)^{-1} = \left(\cosh \frac{ge_0 \mathcal{E}}{2} \right)^{-1} P_{\mathcal{E}} + \left(\cos \frac{ge_0 \mathcal{H}}{2} \right)^{-1} P_{\mathcal{H}}. \tag{B25}$$

We substitute (B24) and (B25) in (B23) to get, by virtue of the properties (A9) of the projection operators, $P_{\mathcal{E}}$ and $P_{\mathcal{H}}$,

$$B - \bar{B} = \frac{1}{ge_0} \left(\cosh \frac{ge_0 \mathcal{E}}{2} \cos \frac{ge_0 \mathcal{H}}{2} \right)^{-1} C. \tag{B26}$$

Finally, inserting (B26) into (B15) and using

$$\bar{\Lambda}(e_0) = \left(\cosh \frac{ge_0 \mathcal{E}}{2} \cos \frac{ge_0 \mathcal{H}}{2} \right)^{-1}, \tag{B27}$$

---

[6]The projection operators $P_{\mathcal{E}}$ and $P_{\mathcal{H}}$ are defined in Appendix A.

one finds that $Q = 0$ and the identity (B12) takes place.

Going to the identity (B13) we use (B12), (15), the identity

$$\gamma^\mu \gamma^5 = -\frac{1}{3!} \epsilon^\mu {}_{\kappa\rho\sigma} \gamma^\kappa \gamma^\rho \gamma^\sigma$$

and the antisymmetry $B_{\alpha\beta} = B_{[\alpha\beta]}$ to bring the LHS into the form

$$
\gamma^\mu \Delta[nx_{cl}, e_0] = \left[\left(\eta^\mu {}_\beta + \frac{ge_0}{2} B^\mu {}_\beta \right) \gamma^\beta \right.
$$
$$
\left. + \frac{ge_0}{4} \left(\eta^\mu {}_\nu B_{\alpha\beta} - \frac{ge_0}{4!} B_{\rho\sigma} B^{*\rho\sigma} \epsilon^\mu {}_{\nu\alpha\beta} \right) \gamma^{[\nu} \gamma^\alpha \gamma^{\beta]} \right] \bar{\Lambda}(e_0). \tag{B28}
$$

¿From (B17) and (B20) one obtains

$$K(1) = \eta + \frac{ge_0}{2} B, \tag{B29}$$

$$K_{\mu[\nu}(1) B_{\alpha\beta]} = \eta_{\mu[\nu} B_{\alpha\beta]} + \frac{ge_0}{2} B_{\mu[\lambda} B_{\alpha\beta]}. \tag{B30}$$

Due to the antisymmetry of B,

$$B_{\mu[\lambda} B_{\alpha\beta]} = B_{[\mu\nu} B_{\alpha\beta]} = -\frac{2}{4!} B_{\kappa\rho} B^{*\kappa\rho} \epsilon_{\mu\nu\alpha\beta},$$

and, substituting in (B30), we get

$$K_{\mu[\nu}(1) B_{\alpha\beta]} = \eta_{\mu[\nu} B_{\alpha\beta]} - \frac{ge_0}{4!} B_{\kappa\rho} B^{*\kappa\rho} \epsilon_{\mu\nu\alpha\beta}. \tag{B31}$$

Finally, we use (B29), (B31) in (B28) to get (B13).

3. Identities involving γ-matrices in $2+1$ dimensions

To prove the identity (106) let us introduce

$$z^\mu = \epsilon^{\mu\nu\lambda} F_{\nu\lambda}, \qquad z^2 = -4\phi^2$$

and transform the LHS of (106) using (96),

$$\exp \left\{ \frac{ge_0}{4} F_{\mu\nu} \gamma^\mu \gamma^\nu \right\} = \cosh \frac{ge_0 \varphi}{2} \left(1 - \frac{iz\gamma}{2\varphi} \tanh \frac{ge_0 \varphi}{2} \right).$$

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Taking into account eqs.(14), (A13) and the relation $iz\gamma = -\gamma F\gamma$, one gets (106).

Multiplying (107) by $K(1)$ and using (96) one transforms (107) into the equivalent identity

$$K^{\mu}{}_{\lambda}(1)\gamma^{\lambda}e^{\frac{ge_0}{4}F_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}} = \gamma^{\mu}\cosh\frac{ge_0\varphi}{2}\det K(1),\qquad(B32)$$

which we are going to prove. Taking into account the identity

$$\gamma^{\lambda}e^{\frac{ge_0}{4}F_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}} = \left(e^{-\frac{ge_0}{2}F}\right)^{\lambda}{}_{\rho}\gamma^{\rho},$$

which can be easily derived from (106) and using (14) one transforms the LHS of (B32),

$$K^{\mu}{}_{\lambda}(1)\gamma^{\lambda}e^{\frac{ge_0}{4}F_{\alpha\beta}\gamma^{\alpha}\gamma^{\beta}} = \left(\cosh\frac{ge_0}{2}\varphi\right)^{-1}\gamma^{\mu},\qquad(B33)$$

Calculating the determinant

$$\det K(1) = \left(\cosh\frac{ge_0\varphi}{2}\right)^{-2},$$

one finds that the RHS of (B33) coincides with that of (B32).

Acknowledgments

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